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Small Quantum Structures with Small State Spaces

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Abstract We summarize and extend results about "small" quantum structures with small dimensions of state spaces. These constructions have contributed to the theory of orthomodular lattices. More general quantum structures (orthomodular posets, orthoalgebras, and effect algebras) admit sometimes simplifications, but there are problems where no progress has been achieved.

Keywords Orthomodular lattice · Orthomodular poset · Orthoalgebra · Effect algebra · State · Probability measure · Group-valued measure

1 Motivation and Basic Notions

In this paper, we study different quantum structures. By these we mean not only Hilbert lattices, but their generalizations, from orthomodular lattices to effect algebras. The development of the theory of orthomodular lattices (OMLs) and orthomodular posets (OMPs) has been encouraged and supported by the constructions of structures admitting no or very few states. We may recall e.g. [4, 28]. Later on, more general structures—orthoalgebras (OAs), lattice effect algebras (LEAs), and effect algebras (EAs)—have been introduced. Sometimes these allow simplifications of the preceding techniques, sometimes no progress has been achieved. (We refer to [3] and [1, 6, 7, 11, 24] for the basic definitions on quantum structures.)

Greechie diagrams have been first introduced in [4] as a tool for a construction of orthomodular lattices admitting no states.

In the sequel, we use the following notation: **1** is the trivial Boolean algebra (with one element), **2** is the two-element Boolean algebra (with one atom), $\mathbf{2}^n$, $n \in \mathbb{N}$, is the Boolean

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algebra with n atoms and 2^n elements, in particular, $\mathbf{2}^1 = \mathbf{2}$, $\mathbf{2}^0 = \mathbf{1}$. By a *state* on a quantum structure we mean a probability measure, i.e., a real non-negative additive functional. (As we shall deal only with small finite structures, the dilemma of finite/countable additivity is irrelevant.) The state space, $S(L)$, of a quantum structure L is the set of all states with the usual affine structure and product topology.

If a quantum structure admits no state, its empty state space can be considered affinely homeomorphic to that of **1**. If a quantum structure has only one state, its state space is affinely homeomorphic to that of **2**. There may be a difference which cannot be described by the affine homeomorphism only. The only state on **2** attains only the values 0 and 1. In contrast to this, one can find a quantum structure (even an orthomodular lattice) admitting only one state such that this state attains also values between 0 and 1. In order to recognize this, we introduce the notion of functional isomorphism.

2 Functional Isomorphism

For a quantum structure L, states can be considered as elements of its dual, $L^* = R^L$, more exactly, of $[0, 1]^L$. There is a natural embedding $\mathbf{e}$ of L into its second dual, $L^{**} = R^{L^*}$, more exactly, into $[0, 1]^{S(L)}$, defined by

$$\mathbf{e}(a)(s) = s(a) \quad \text{for all } a \in L,\ s \in S(L).$$

The functional $\mathbf{e}(a)\colon S(L) \to [0, 1]$ is called the *evaluation functional* associated with a. We use the notation $\mathbf{e}(L) = \{\mathbf{e}(a) : a \in L\}$ for the collection of all evaluation functionals of L.

A functional isomorphism is a correspondence of sets of evaluation functionals of two quantum structures which assumes that the state spaces are affinely homeomorphic. (It can be formulated in a more general context, cf. [17].)

Definition 1 *Let L_1, L_2 be effect algebras. A mapping $g\colon \mathbf{e}(L_1) \to \mathbf{e}(L_2)$ is called a* functional isomorphism *iff it is one-to-one and there is an affine homeomorphism $h\colon S(L_1) \to S(L_2)$ such that*

$$[f_2 = g(f_1),\ s_2 = h(s_1)] \Longrightarrow f_2(s_2) = f_1(s_1)\,.$$

If there is a functional isomorphism $g\colon \mathbf{e}(L_1) \to \mathbf{e}(L_2)$, then L_1, L_2 are called functionally isomorphic.

The importance of the functional isomorphism follows from the fact that it preserves many properties of state spaces, but it allows to represent some complex structures by much simpler ones which are functionally isomorphic. This simplification occurs when different elements have the same evaluation functionals. The affine homeomorphism between state spaces is often derived from a simple restriction mapping.

Of course, the smallest quantum structures whose state spaces are affinely homeomorphic to that of **1**, resp. **2**, are these Boolean algebras themselves. The reason why we ask of *non-trivial* (i.e., non-Boolean) examples is that further constructions require several atoms on which states behave equivalently. This can be recognized by the evaluation functionals; we need sets of atoms whose evaluation functionals coincide.

We are further interested in (non-trivial) quantum structures which are functionally isomorphic to $\mathbf{2}^n, n \in \mathbb{N}$. Such structures exist even among orthomodular lattices for any $n \in \mathbb{N}$ [19]. To demonstrate their importance, the results of [14, 16, 19, 22] are based on the existence of arbitrarily large OMLs functionally isomorphic to $\mathbf{2}^n$, $n \geq 3$.

3 States on Products

States on products of orthomodular posets are characterized, e.g., in [12]. This result applies to products of other quantum structures as well. As a consequence, we have the following fact; for generality, we formulate it for effect algebras.

Proposition 1 *Let L be an effect algebra admitting no states and let* $n \in \mathbb{N}$. *Then the product* $L \times \mathbf{2}^n$ *is functionally isomorphic to* $\mathbf{2}^n$; *in particular, the state spaces* $S(L \times \mathbf{2}^n)$, $S(\mathbf{2}^n)$ *are affinely homeomorphic.*

For $n = 1$, we obtain a quantum structure with exactly one state. This observation has been made in [23].

Notice that, replacing L with $L \times \mathbf{2}^n$, the number of blocks (= maximal subalgebras of mutually compatible elements) is preserved, the number of atoms is increased by n, the number of elements is multiplied by 2^n.

Sometimes this construction is smaller than the concurrent specific constructions of quantum structures with the same state space.

4 Lattice Effect Algebras

Lattice effect algebras, i.e., effect algebras which are lattices, form a specific type of quantum structures. They were deeply studied, e.g., in [25]. As follows from [10, 25, 27], the states on complete (in particular, finite) LEAs correspond uniquely to states on the subalgebra of *sharp elements*. As a consequence, the state space of a finite LEA is affinely homeomorphic to a smaller OML. Thus LEAs do not contribute to the constructions of the least quantum structures with given state space properties and we will not deal with them in the sequel.

5 Quantum Structures Admitting no States

Quantum structures admitting no states are called *exotic* in [23].

The first example of an OML admitting no states is a famous result of [4]. It has been immediately used by Shultz in the proof that every compact convex set is affinely homeomorphic to the state space of some OML [28]. This proof has been considerably simplified using functionally isomorphic OMLs in [16, 19]. Thus an originally negative result (a specific counterexample) allowed to prove an interesting positive result (existence of OMLs with given properties). This predicted also the future development. The Shultz' result has been largely generalized: We can find an OML with a given (compact convex) state space, but also with a given centre, automorphism group, and subalgebra (admitting states), see [9, 15, 20].

Shultz used a different basic construction than that of [4]; based on it, Rogalewicz has found another OML admitting no states; it is not the smallest known, but it possesses nice symmetry. The smallest known OML admitting no states has been constructed by Mayet [13] and yet unpublished, see Fig. 1. This example is in some sense optimal and it seems that there is no smaller example [21].

There is a simple example of an OMP admitting no states in Fig. 2. Surprisingly, no simpler orthoalgebra with this property is known.

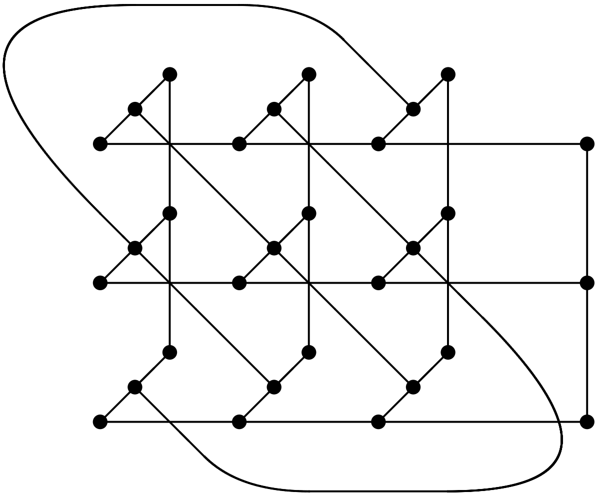

Fig. 1 Mayet's example of an OML without states

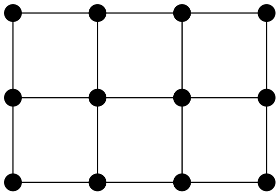

Fig. 2 OMP without states

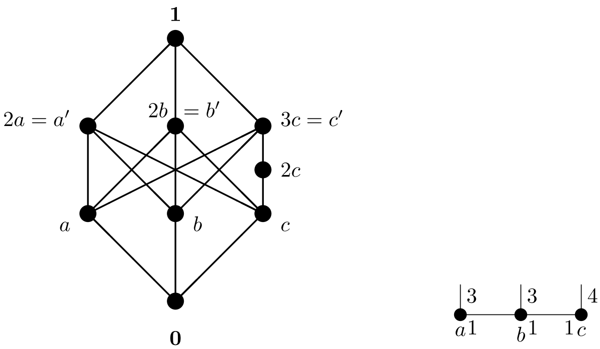

Fig. 3 Hasse and (generalized) Greechie diagram of Riecanová's example of an EA without states

The smallest known effect algebra admitting no states is represented by the Hasse and (generalized) Greechie diagram in Fig. 3. It has been described by Riečanová in [26].

A question has been open for some time whether there are quantum structures admitting even no non-zero group-valued measure (with values in any commutative group). A positive answer—even for OMLs—has been proved in [18] and independently in [29]. Both techniques needed computer proofs, although the former allows also a relatively easy verification. We are not aware of any attempts to find simpler examples among more general quantum structures.

The complexity of examples of quantum structures admitting no states is compared in Table 1.

Table 1 Size of the least known examples

State space like	Type of QS	Functionally isomorphic	Example	No. of atoms	No. of blocks	No. of elements
1	OML	YES	[4]	36	23	92
		YES	Mayet	30	19	80
		YES	Rogalewicz web	39	25	98
	OMP, OA	YES	folklore	12	7	44
	EA	YES	[26]	3	4	9
no group-valued	OML	YES	[18]	264	270	548
		YES	[29]	74	78	156
2	OML	NO	[18] tire 44	44	44	90
		YES	Mayet and [23]’	31	19	160
	OMP	NO	[5] tire 22	22	22	46
		YES	[23]	13	7	88
	OA	NO	Fano plane	7	7	16
	EA	NO	diamond	2	2	4
	MVA, EA	NO	$\mathcal{L}_2$	1	1	3
$\mathbf{2}^2$	OML	NO	tire 44’	44	43	90
		YES	Mayet and [23]’	32	19	320
	OMP	NO	tire 22’	22	21	46
		YES	[23]’	14	7	176
	OA, EA	NO	[8] Fano plane’	7	6	16
$\mathbf{2}^3$	OML	YES	tire 66	66	64	134
	OMP	YES	tire 18	18	16	38
	OA, EA	YES	folklore	6	4	14

6 Quantum Structures with Exactly One State

Quantum structures admitting exactly one state are called *nearly exotic* in [23]. As observed there, examples of quantum structures with exactly one state can be constructed from quantum structures admitting no state using Proposition 1. The states obtained this way are necessarily two-valued (attaining only values 0 and 1). There are specific constructions leading sometimes to simpler results and to states which are not necessarily two-valued.

As a preliminary step, an OML with exactly one state has been introduced in [18]. It has 44 atoms, $a_0, \ldots, a_{43}$ and 44 blocks, with sets of atoms $\{a_{2k}, a_{2k+1}, a_{2k+2}\}$ and $\{a_{2k-7}, a_{2k}, a_{2k+13}\}$, where $k = 0, \ldots, 21$ and indices are evaluated modulo 44. Its only state attains $1/3$ at each atom. Its Greechie diagram (the "*tire 44*") is shown in Fig. 4 only for curiosity. In order to distinguish the blocks, the edges do not touch their ending vertices (because other edges end there from the other side). An OMP with this property has been constructed long before in [5] (the "*tire 22*"). It has 22 atoms, $a_0, \ldots, a_{21}$ and 22 blocks, with sets of atoms $\{a_{2k}, a_{2k+1}, a_{2k+2}\}$ and $\{a_{2k-5}, a_{2k}, a_{2k+5}\}$, where $k = 0, \ldots, 11$ and indices are evaluated modulo 22.

When these two examples are compared to the technique of Proposition 1 and results of Sect. 5, we obtain examples with less blocks and atoms, but more elements.

The smallest known orthoalgebra with exactly one state is the *Fano plane*, see the Greechie diagram in Fig. 5.

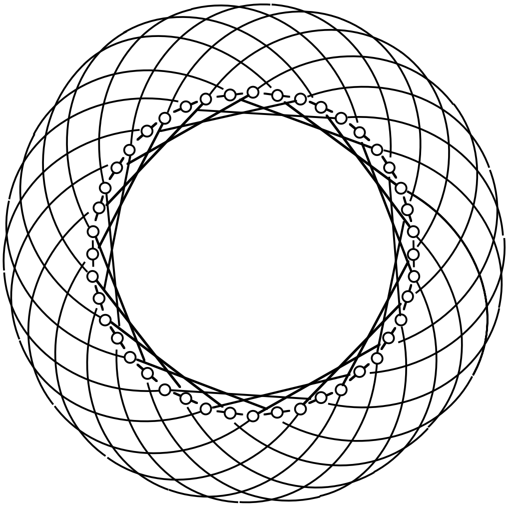

Fig. 4 OML with one state

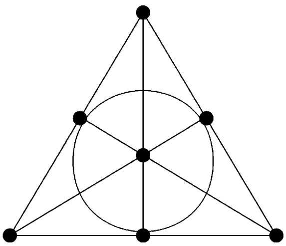

Fig. 5 OA with one state (Fano plane)

Fig. 6 Hasse and (generalized) Greechie diagram of an MV-algebra with one state

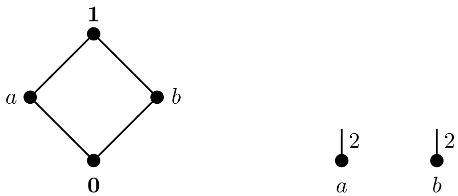

Fig. 7 Hasse and (generalized) Greechie diagram of an EA with one state (diamond)

The smallest non-Boolean effect algebra with exactly one state is the three-element chain with the structure of an MV-algebra, called $\mathcal{L}_2$ in [2], see the Hasse and (generalized) Greechie diagrams in Fig. 6. The smallest effect algebra with exactly one state and *non-compatible elements* is the *diamond*, which is the horizontal sum of two copies of $\mathcal{L}_2$, see the Hasse and (generalized) Greechie diagrams in Fig. 7.

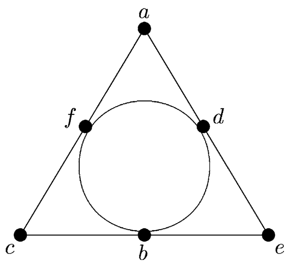

Fig. 8 OA functionally isomorphic to $\mathbf{2}^3$

7 Quantum Structures Functionally Isomorphic to $\mathbf{2}^2$

Examples of quantum structures functionally isomorphic to $\mathbf{2}^2$ can be constructed using Proposition 1. The product of $\mathbf{2}^2$ and a quantum structure admitting no state is functionally isomorphic to $\mathbf{2}^2$. There are specific constructions leading sometimes to simpler results. Very often it suffices to omit one block in a quantum structure with exactly one state and we obtain a structure admitting a one-dimensional state space. However, these structures usually are not functionally isomorphic to $\mathbf{2}^2$ because some elements correspond to other evaluation functionals. Usually the range of an evaluation functional is not the whole interval [0, 1].

8 Quantum Structures Functionally Isomorphic to $\mathbf{2}^3$

Examples of quantum structures functionally isomorphic to $\mathbf{2}^3$ can be constructed from quantum structures admitting no state using Proposition 1. There are specific constructions leading sometimes to simpler results.

Such an OML has been introduced in [16] (the "*tire 66*"). It has 66 atoms, $a_0, \ldots, a_{65}$ and 66 blocks, with sets of atoms $\{a_{2k}, a_{2k+1}, a_{2k+2}\}$ and $\{a_{2k-7}, a_{2k}, a_{2k+13}\}$, where $k = 0, \ldots, 32$ and indices are evaluated modulo 66. (Two blocks are unnecessary and they are added only for symmetry.) An OMP with this property has been constructed long before in [14] (the "*tire 18*"). It has 18 atoms, $a_0, \ldots, a_{18}$ and 18 blocks, with sets of atoms $\{a_{2k}, a_{2k+1}, a_{2k+2}\}$ and $\{a_{2k-5}, a_{2k}, a_{2k+5}\}$, where $k = 0, \ldots, 8$ and indices are evaluated modulo 18. (Two blocks are unnecessary.)

When these two examples are compared to the technique of Proposition 1 and results of Sect. 5, we obtain examples with less blocks and atoms, but more elements.

The smallest known orthoalgebra functionally isomorphic to $\mathbf{2}^3$ is represented by the Greechie diagram in Fig. 8. No smaller effect algebra with this property is known.

9 "Book of Records"

The complexity of examples of (non-trivial) quantum structures with small state spaces is compared in Table 1. We always require affine homeomophism between state spaces; if even a functional isomorphism is achieved, it is mentioned explicitly. References to sources are given where available. Primes denote constructions obtained by slight modifications of the original examples.

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References

1. Beran, L.: Orthomodular Lattices. Algebraic Approach. Academia, Praha (1984)
2. Cignoli, R., D'Ottaviano, I.M.L., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. Kluwer Academic, Dordrecht (1999)
3. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer Academic, Dordrecht & Ister, Bratislava (2000)
4. Greechie, R.J.: Orthomodular lattices admitting no states. J. Comb. Theory Ser. A **10**, 119–132 (1971)
5. Greechie, R.J., Miller, F.R.: On structures related to states on an empirical logic I. Weights on finite spaces. Technical Report 16, Department of Mathematics, Kansas State University, Manhattan, KS (1970)
6. Gudder, S.P.: Quantum Probability. Academic, New York (1988)
7. Hamhalter, J.: Quantum Measure Theory. Kluwer Academic, Dordrecht (2003)
8. Hamhalter, J., Navara, M., Pták, P.: States on orthoalgebras. Int. J. Theor. Phys. **34**(8), 1439–1465 (1995)
9. Harding, J., Navara, M.: Embeddings into orthomodular lattices with given centers, state spaces and automorphism groups. Order **17**(3), 239–254 (2000)
10. Jenča, G.: The block structure of complete lattice ordered effect algebras. J. Austral. Math. Soc. (2007, to appear)
11. Kalmbach, G.: Orthomodular Lattices. Academic, London (1983)
12. Maňasová, V., Pták, P.: On states on the product of logics. Int. J. Theor. Phys. **20**, 451–457 (1981)
13. Mayet, R.: Personal communication (1993)
14. Navara, M.: State space properties of finite logics. Czechoslov. Math. J. **37**(112), 188–196 (1987)
15. Navara, M.: Independence of automorphism group, center and state space of quantum logics. Int. J. Theor. Phys. **31**, 925–935 (1992)
16. Navara, M.: Descriptions of state spaces of orthomodular lattices. Math. Bohem. **117**, 305–313 (1992)
17. Navara, M.: Two descriptions of state spaces of orthomodular structures. Int. J. Theor. Phys. **38**(12), 3163–3178 (1999)
18. Navara, M.: An orthomodular lattice admitting no group-valued measure. Proc. Am. Math. Soc. **122**, 7–12 (1994)
19. Navara, M.: State spaces of quantum structures. Rend. Ist. Mat. Univ. Trieste **31**(1), 143–201 (2000)
20. Navara, M., Pták, P., Rogalewicz, V.: Enlargements of quantum logics. Pacific J. Math. **135**, 361–369 (1988)
21. Navara, M., Riečanová, Z.: Existence of states on quantum structures. Preprint (2007)
22. Navara, M., Rogalewicz, V.: Construction of orthomodular lattices with given state spaces. Demonstr. Math. **21**, 481–493 (1988)
23. Pták, P.: Exotic logics. Coll. Math. **54**, 1–7 (1987)
24. Pták, P., Pulmannová, S.: Orthomodular Structures as Quantum Logics. Kluwer Academic, Dordrecht (1991)
25. Riečanová, Z.: Generalization of blocks for D-lattices and lattice ordered effect algebras. Int. J. Theor. Phys. **39**, 231–237 (2000)
26. Riečanová, Z.: Proper effect algebras admitting no states. Int. J. Theor. Phys. **40**, 1683–1691 (2001)
27. Riečanová, Z.: Smearings of states defined on sharp elements onto effect algebras. Int. J. Theor. Phys. **41**, 1511–1524 (2002)
28. Shultz, F.W.: A characterization of state spaces of orthomodular lattices. J. Comb. Theory A **17**, 317–328 (1974)
29. Weber, H.: There are orthomodular lattices without non-trivial group valued states; a computer-based construction. J. Math. Anal. Appl. **183**, 89–94 (1994)